

Viscous Dissipation in 2D Fluid Dynamics as a Symplectic Process

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Abstract. Dissipation can be represented in Hamiltonian mechanics in an extended phase space. The method uses an unphysical auxiliary variable which represents the excitation of unresolved dynamics (subscales) and a Hamiltonian for the interaction between the resolved dynamics and the auxiliary variable. This method is applied to viscous dissipation (including hyper-viscosity) in a two-dimensional fluid, for which the dynamics is non-canonical. We derive a metriplectic representation and suggest a measure for the entropy of the system.

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1. Introduction

Physical processes that conserve the total energy can be described in a Hamiltonian representation, in which the phase space volume is conserved. Hamiltonian dynamics is represented by a symplectic Poisson bracket which conserves a Hamiltonian function and satisfies Liouville's Theorem. If the bracket is nonsingular, the Hamiltonian is a conserved quantity and the structure is also denoted as canonical. Otherwise, if the bracket is singular, the dynamics is called noncanonical and additional conservation laws exist, that take the name of Casimir functions. A classic example of singular Poisson brackets is given by fluid dynamics [1], where there are infinitely many Casimir functions, two notable examples of which are given by the enstrophy and the helicity.

Dissipative processes decrease kinetic energy and contract the volume in phase space. There are two common approaches to embed dissipative processes in a geometric framework [2]: *gradient* systems and *metriplectic* systems. Gradient systems, also called incomplete systems, represent the dynamics by the gradient of a potential which is clearly not divergence-free. They dissipate energy and, possibly, other physical quantities and, by Lyapunov's theorem, have built-in asymptotic stability. A famous example of a gradient system is given by the Navier-Stokes equations. A special case of a gradient system applied to fluid dynamics and geophysical fluid dynamics is given by [3, 4, 5], in which dissipation of energy and conservation of a Casimir (either helicity or enstrophy depending if the system is 2D or 3D) is used to find exact solutions of the Euler equations without changing the topology of the flow.

Metriplectic systems, also called complete systems, are combinations of a symplectic and a metric bracket. Metriplectic systems are formulated by a noncanonical symplectic bracket with an entropy-like Casimir [2], and a metric bracket describing diffusion. Heuristically, the relationship between the Casimirs and the entropy in fluid dynamics is given by the fact that the Casimirs are associated to a relabeling symmetry [6, 7], and hence to a counting of states [2]. The symplectic Poisson bracket conserves the Hamiltonian and entropy while a metric bracket preserves the Hamiltonian and increases entropy. The conservation of energy and the increase of entropy are representation of the first and second laws of thermodynamics respectively. Metriplectic systems are typical in standard kinetic equations [8, 9, 10, 11, 12, 13, 14]. For an example applied to geophysical fluid dynamics see [15].

Lagrangian and Hamiltonians mechanics is traditionally not able to represent dissipative processes like friction or viscosity in terms of macroscopic variables. To circumvent this limitation a method based on the extension of phase space was suggested in [16], see also [17, 18]. The same method was used, for example, in [19] for an application to quantum diffusion. See also [20, 21] for reviews of different approaches. The main ingredient proposed by [16] is the introduction of an auxiliary variable which grows in the time-reversed model with negative friction or, in the case of fluid dynamics, negative viscosity. The method thus consists in finding the adjoint equation of the system [17]. This auxiliary variable has no predefined physical meaning and in this

study we will identify it with the excitation of unresolved (also called subgrid or subscale) processes.

Here we use the interaction Hamiltonian of [16] for the dissipative interaction between the macroscopic variables and the auxiliary variable. To introduce the method we first consider the finite dimensional example of solid body friction. The method is then applied to viscosity in 2D fluid dynamics.

The interaction Hamiltonian has the form $H = w\chi$ for the macroscopic variable w and an auxiliary degree of freedom χ , which yields unbounded hyperbolic trajectories in phase space. In a canonical bracket, this interaction Hamiltonian leads to irreversible decay and growth. This can be compared with the so-called $H = xp$ Hamiltonian which has attracted particular interest, as it is conjectured to have a relationship with the zeros of the Riemann function, see e.g. [22, 23, 24]. Note that the interaction Hamiltonian is unrelated to the conservative dynamics.

The paper is organized as follows: In Section 2 we consider friction of two bodies with different velocities as an example with an interaction Hamiltonian in a canonical Hamiltonian framework. In Section 3 viscous diffusion in a two-dimensional incompressible fluid is embedded in a symplectic bracket, which is added to the noncanonical Hamiltonian governing ideal dynamics. The viscous diffusion will include both the cases of Rayleigh friction and of hyperviscosities, which are commonly applied in numerical models. Section 4 summarizes and discusses the main results of the study.

2. An Introductory Finite Dimensional Example: Friction at an Interface

As an introductory example, we consider the friction at the interface between two solid bodies moving with velocities u and v . We consider friction from a purely phenomenological point of view without reference to microscopic processes at the boundaries of the two bodies. For a review on the theoretical modelling of sliding friction see [25]. The most simple model of friction is the linear decay with a coefficient γ

$$\dot{u} = \gamma(v - u), \quad \dot{v} = \gamma(u - v) . \quad (1)$$

While the mean $U = (u + v)/2$ remains constant, the dynamics is given by the velocity shear $w = (u - v)/2$, which decays according to

$$\dot{w} = -2\gamma w , \quad (2)$$

with the solution $w = w_0 \exp(-2\gamma t)$. To embed this in a symplectic framework we extend the phase space by a second, auxiliary variable χ and apply the method of [16] (pp. 298-299) for an overdamped harmonic oscillator. We assume that χ describes the excitation of subscale degrees of freedom, without further physical interpretation. The auxiliary variable follows a time reversed dynamics and grows for a frictional decay of w . The method thus consists in finding the adjoint of (2). We construct a phase

space by the two degrees of freedom w and χ . To satisfy Liouville's Theorem

$$\frac{\partial \dot{w}}{\partial w} + \frac{\partial \dot{\chi}}{\partial \chi} = 0 , \quad (3)$$

we assume that the auxiliary variable compensates the decay of w and increases continuously

$$\frac{\partial \dot{w}}{\partial w} = -2\gamma , \quad (4)$$

$$\frac{\partial \dot{\chi}}{\partial \chi} = +2\gamma , \quad (5)$$

with the solution

$$\dot{\chi} = +2\gamma\chi , \quad \chi = \chi_0 \exp(+2\gamma t) . \quad (6)$$

It should be noted that the first equation of (6) is the adjoint of (2). Liouville's Theorem (3) can be used to introduce a canonical Hamiltonian system for the friction process

$$\dot{w} = -2\gamma \frac{\partial M}{\partial \chi} , \quad \dot{\chi} = 2\gamma \frac{\partial M}{\partial w} , \quad (7)$$

with an interaction Hamiltonian

$$M = w\chi . \quad (8)$$

This Hamiltonian was introduced by [16] for the overdamped harmonic oscillator. The trajectories induced by M are hyperbolic and unbounded, so that friction is represented as an irreversible process.

Note that the solution found here is specific under the request of the validity of Liouville's theorem. For a general discussion of the Liouville equations for non-Hamiltonian systems, see [26] and references therein. The use of Liouville's theorem will be particularly useful in the discussion of fluid dynamics systems in Nambu form in the next Section, which is solely based on the conservation of phase space volume [27, 28].

The Hamiltonian system presented in this Section can as well be represented as a gradient system

$$\dot{w} = 2\gamma \frac{\partial \phi}{\partial w} , \quad \dot{\chi} = 2\gamma \frac{\partial \phi}{\partial \chi} , \quad (9)$$

with the potential

$$\phi = \frac{1}{2} (\chi^2 - w^2) , \quad (10)$$

and where $-\gamma w^2/2$ is the Rayleigh dissipation function. The potential ϕ grows as

$$\dot{\phi} = -w\dot{w} + \chi\dot{\chi} = 2\gamma (w^2 + \chi^2) > 0 . \quad (11)$$

Using the unit vector perpendicular to the w - χ -plane, $\vec{k} = (0, 0, 1)$, we see that the contours of M and ϕ are orthogonal

$$(\dot{w}, \dot{\chi}) = \vec{k} \times \vec{\nabla} M = \vec{\nabla} \phi . \quad (12)$$

The two equations given by (12) are the Cauchy-Riemann equations for the analytic complex function

$$f(z) = z^2/2 = -\phi + iM , \quad (13)$$

for the argument $z = w + i\chi$ (see also [29]).

After this preliminary example, we will consider the representation of dissipation in an infinite dimensional system, represented by two-dimensional fluid dynamics.

3. Two-dimensional fluid dynamics with viscous diffusion

Two-dimensional incompressible fluid dynamics with viscous diffusion can be described by the vorticity equation

$$\frac{\partial \omega}{\partial t} = J(\psi, \omega) + \nu \nabla^2 \omega . \quad (14)$$

The Jacobian $J(a, b) = a_x b_y - a_y b_x$ is used to represent the advection of the vorticity $\omega = \nabla^2 \psi$ by the flow, $(u, v) = (-\partial_y \psi, \partial_x \psi)$, where ψ is the stream-function. In (14) ν is the (kinematic) viscosity. The vorticity equation can be described as a noncanonical Hamiltonian system [7]

$$\frac{\partial \omega}{\partial t} = \{\omega, \mathcal{H}\}_{\mathcal{E}} , \quad (15)$$

with the Hamiltonian in a periodic domain given by the kinetic energy

$$\mathcal{H} = \frac{1}{2} \int (\nabla \psi)^2 dA = - \int \psi \omega dA . \quad (16)$$

The noncanonical Poisson bracket

$$\{\mathcal{F}, \mathcal{H}\}_{\mathcal{E}} = \int \omega J \left(\frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{H}}{\delta \omega} \right) dA \quad (17)$$

has enstrophy \mathcal{E} as a Casimir

$$\mathcal{E} = \frac{1}{2} \int \omega^2 dA , \quad (18)$$

since $\{\mathcal{F}[\omega], \mathcal{E}\}_{\mathcal{E}} = 0$ for any functional $\mathcal{F}[\omega]$. This Casimir follows from the particle relabeling symmetry [6, 7]. In the following, the bracket (17) will be called advective bracket.

To rewrite the viscous diffusion of ω , we introduce an interaction functional analogous to (8)

$$\mathcal{M} = \int \omega \sigma dA , \quad (19)$$

with an auxiliary variable σ . This variable σ is an unspecified field representing local subscale processes. We define an anti-symmetric bracket for two functionals \mathcal{V} and \mathcal{W} depending on ω and σ

$$\{\mathcal{V}, \mathcal{W}\}' = \nu \int [\mathcal{V}_{\omega} \nabla^2 \mathcal{W}_{\sigma} - \mathcal{V}_{\sigma} \nabla^2 \mathcal{W}_{\omega}] dA , \quad (20)$$

where

$$\mathcal{V}_\omega = \frac{\delta \mathcal{V}}{\delta \omega}, \quad \mathcal{W}_\omega = \frac{\delta \mathcal{W}}{\delta \omega}, \quad \mathcal{V}_\sigma = \frac{\delta \mathcal{V}}{\delta \sigma}, \quad \mathcal{W}_\sigma = \frac{\delta \mathcal{W}}{\delta \sigma}, \quad (21)$$

denote functional derivatives. The bracket (20) is anti-symmetric due to the symmetry of ∇^2 ,

$$\{\mathcal{V}, \mathcal{W}\}' = -\{\mathcal{W}, \mathcal{V}\}' . \quad (22)$$

Since ω and σ are independent degrees of freedom we have

$$\{\omega, \sigma\}' = 0 . \quad (23)$$

It should be noted that, although (20) is used to describe dissipation, it is not a metric bracket. The viscous diffusion term for the vorticity is obtained through this bracket $\{\cdot, \cdot\}'$ and the functional \mathcal{M} , so that

$$\left. \frac{\partial \omega}{\partial t} \right|_{visc} = \{\omega, \mathcal{M}\}' = \nu \nabla^2 \omega . \quad (24)$$

To distinguish $\{\cdot, \cdot\}'$ from the Poisson bracket (17) we denote it as diffusive bracket. A Poisson bracket satisfies two conditions, the Leibniz rule (or derivation property) and the Jacobi identity. It can be proven that the diffusive bracket satisfies the Leibniz rule which is similar to the product rule for differentiation,

$$\{\mathcal{F}\mathcal{G}, \mathcal{W}\}' = \mathcal{F}\{\mathcal{G}, \mathcal{W}\}' + \mathcal{G}\{\mathcal{F}, \mathcal{W}\}' , \quad (25)$$

for arbitrary functionals \mathcal{F} , \mathcal{G} and \mathcal{W} . The Jacobi identity, however, is not satisfied

$$\{\mathcal{F}, \{\mathcal{G}, \mathcal{W}\}'\}' + \{\mathcal{G}, \{\mathcal{W}, \mathcal{F}\}'\}' + \{\mathcal{W}, \{\mathcal{F}, \mathcal{G}\}'\}' \neq 0 , \quad (26)$$

as can be seen by simple substitution. Therefore, the diffusive bracket is not a Poisson-bracket.

The auxiliary variable σ grows like a 'negative viscosity'

$$\frac{\partial \sigma}{\partial t} = \{\sigma, \mathcal{M}\}' = -\nu \nabla^2 \sigma . \quad (27)$$

In this framework the viscous decay of any functional $\mathcal{F}[\omega]$ is determined by the bracket $\{\mathcal{F}, \mathcal{M}\}'$. The kinetic energy, given by the Hamiltonian \mathcal{H} , decays as

$$\frac{d\mathcal{H}}{dt} = \{\mathcal{H}, \mathcal{M}\}' = -\nu \int \mathcal{H}_\omega \nabla^2 \mathcal{M}_\sigma dA , \quad (28)$$

since \mathcal{H} does not depend on σ . This decay is proportional to the enstrophy

$$\frac{d\mathcal{H}}{dt} = -\nu \int \psi \nabla^2 \omega dA = -\nu \int \omega^2 dA = -2\nu \mathcal{E} . \quad (29)$$

The enstrophy decays as

$$\{\mathcal{E}, \mathcal{M}\}' = \nu \int \omega \nabla^2 \omega dA \leq 0 , \quad (30)$$

since the eigenvalues of ∇^2 are negative in a periodic domain. In the numerical modelling of geophysical flows, instead of viscosity, hyper-viscosity (or hyper-diffusion)

is frequently used which, compared to (14), is concentrated at high wave numbers with a reduced impact at low wave numbers

$$\left. \frac{\partial \omega}{\partial t} \right|_{hyp-visc} = (-1)^{(n+1)} \nu \nabla^{2n} \omega . \quad (31)$$

Equation (31) includes linear friction $-\nu\omega$ for $n = 0$. Hyper-viscosity is represented by a similar bracket as (20), with the same functional \mathcal{M}

$$\{\mathcal{F}, \mathcal{M}\}'' = (-1)^{(n+1)} \nu \int [\mathcal{F}_\omega \nabla^{2n} \mathcal{M}_\sigma - \mathcal{F}_\sigma \nabla^{2n} \mathcal{M}_\omega] dA . \quad (32)$$

The bracket $\{\mathcal{F}, \mathcal{M}\}'$ can be separated into two terms by replacing \mathcal{M} by two independent integrals, the enstrophy \mathcal{E} and the integral of the square of σ ,

$$\mathcal{E} = \frac{1}{2} \int \omega^2 dA , \quad \mathcal{B} = \frac{1}{2} \int \sigma^2 dA , \quad (33)$$

If we replace the functional derivatives of \mathcal{M} by those of \mathcal{E} and \mathcal{B} ,

$$\mathcal{E}_\omega = \mathcal{M}_\sigma , \quad \mathcal{B}_\sigma = \mathcal{M}_\omega , \quad (34)$$

we obtain two symmetric brackets which involve derivatives of ω and σ only

$$\begin{aligned} \{\mathcal{F}, \mathcal{M}\}' &= \nu \int [\mathcal{F}_\omega \nabla^2 \mathcal{E}_\omega - \mathcal{F}_\sigma \nabla^2 \mathcal{B}_\sigma] dA \\ &= \nu \int \mathcal{F}_\omega \nabla^2 \mathcal{E}_\omega dA - \nu \int \mathcal{F}_\sigma \nabla^2 \mathcal{B}_\sigma dA \\ &= \langle \mathcal{F}, \mathcal{E} \rangle_\omega + \langle \mathcal{F}, \mathcal{B} \rangle_\sigma . \end{aligned} \quad (35)$$

The two functionals \mathcal{E} and \mathcal{B} split the dynamics in a decaying direction (for ω) and an expanding direction (for σ). The geometric structure changes qualitatively since an anti-symmetric bracket is replaced by two symmetric brackets. This decomposition is comparable to the decomposition of a Nambu bracket by so-called constitutive conservation laws [30, 31, 32]. The ω -bracket yields diffusion,

$$\left. \frac{\partial \omega}{\partial t} \right|_{visc} = \langle \omega, \mathcal{E} \rangle_\omega = \nu \nabla^2 \omega , \quad (36)$$

and the σ -bracket yields the growth of the auxiliary variable σ

$$\frac{\partial \sigma}{\partial t} = \langle \sigma, \mathcal{B} \rangle_\sigma = -\nu \nabla^2 \sigma . \quad (37)$$

Equations (36)-(37) are similar to the representation of the canonical Hamiltonian system as a gradient system (9). The difference $\Phi = \mathcal{B} - \mathcal{E}$ is the analogue of the potential ϕ in (10) and grows according to

$$\frac{d\Phi}{dt} = \{\Phi, \mathcal{M}\}' = \{\mathcal{B}, \mathcal{M}\}' - \{\mathcal{E}, \mathcal{M}\}' \geq 0 . \quad (38)$$

The dynamics of an arbitrary functional \mathcal{F} of vorticity ω and the auxiliary variable σ is given by the non-canonical Poisson bracket (17) and the diffusive bracket

$$\frac{\partial}{\partial t} \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}_\mathcal{E} + \{\mathcal{F}, \mathcal{M}\}' . \quad (39)$$

The main properties of the two brackets are: (i) The advective bracket preserves all integrals

$$\{\mathcal{H}, \mathcal{H}\}_{\mathcal{E}} = 0, \quad \{\Phi, \mathcal{H}\}_{\mathcal{E}} = 0, \quad (40)$$

with the conservation laws

$$\{\mathcal{E}, \mathcal{H}\}_{\mathcal{E}} = 0, \quad \{\mathcal{B}, \mathcal{H}\}_{\mathcal{E}} = 0. \quad (41)$$

And (ii), the diffusive bracket impacts all integrals

$$\{\mathcal{H}, \mathcal{M}\}' \leq 0, \quad \{\Phi, \mathcal{M}\}' \geq 0, \quad (42)$$

and

$$\{\mathcal{E}, \mathcal{M}\}' = \{\mathcal{E}, \mathcal{E}\}_{\omega} \leq 0, \quad \{\mathcal{B}, \mathcal{M}\}' = \{\mathcal{B}, \mathcal{E}\}_{\sigma} \geq 0. \quad (43)$$

It is possible to derive a metriplectic representation based on (39) if the auxiliary variable is disregarded and the dynamics is restricted to the vorticity ω

$$\frac{\partial}{\partial t} \mathcal{F} = \{\mathcal{F}, \mathcal{H}\}_{\mathcal{E}} + \langle \mathcal{F}, \mathcal{E} \rangle_{\omega}. \quad (44)$$

This representation uses a symplectic Poisson-bracket and a metric bracket defined in (35). The role of the enstrophy is thus two-fold: it is a Casimir of the Poisson-bracket and determines the dissipation in the metric bracket. This is the formulation of metriplectic systems suggested by [8]. However, the Hamiltonian (the kinetic energy) is not preserved in the diffusive bracket. This is similar to the decay of the Hamiltonian found by [15] in the metriplectic form of Rayleigh-Bénard convection with viscosity and temperature diffusion.

It should be noted that in the calculations above, the entropy has not been defined. This is one of the disadvantages of metriplectic systems in comparison to gradient systems, where instead dissipation can be linked to Boltzmann's H-Theorem [2]. However, as mentioned in the Introduction, in metriplectic systems the Casimir functions can be considered as candidates for entropy [2], due to the fact that the Casimirs are associated to a relabeling symmetry and hence to a counting of states. The metriplectic representation (44) suggests to consider the (negative) enstrophy as entropy. In the extended phase space formulation this role might be attributed to Φ .

4. Summary and Discussion

In this paper we reconsider the description of dissipative processes based on an extended phase space [16]. An auxiliary variable is added which represents sub scale degrees of freedom and evolves according to time-reversed equations. Further, an interaction Hamiltonian yielding unbounded hyperbolic trajectories in the canonical phase space, which we consider as a necessary property of irreversible dissipative processes, is considered. The dynamics of the extended system is nondivergent in phase space and satisfies Liouville's Theorem. The resulting formulation of dynamics yields thus a symplectic description of dissipation. We have considered two applications

of this approach: a finite dimensional system, where dissipation takes the form of a simple friction of two adjacent bodies moving with different velocities, and an infinite dimensional system, where dissipation takes the form of viscosity in two-dimensional fluid dynamics (with an extension to hyperviscosity or hyperdiffusion). From a geometric point of view, the system considered here is neither an incomplete gradient nor a complete metriplectic system.

The symplectic description of viscosity uses an interaction Hamiltonian given by the bilinear coupling of vorticity and the auxiliary variable. An anti-symmetric bracket (which is not a Poisson bracket) reveals diffusion for the vorticity and 'anti-diffusion' (e.g. a negative viscosity) for the auxiliary variable. By a redefinition of the interaction Hamiltonian this bracket can be split into two symmetric brackets, giving rise to a metriplectic system. In this case, enstrophy appears as a Casimir of the non-canonical bracket and in the symmetric bracket which is responsible for dissipation. This suggests to consider the negative enstrophy as a measure for entropy.

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